Asymptotic algebra for charged particles and radiation

Andrzej Herdegen *
II. Institute of Theoretical Physics, Hamburg University, Luruper Chaussee 149, 22761 Hamburg, Germany

Abstract

A C^*-algebra of asymptotic fields which properly describes the infrared structure in quantum electrodynamics is proposed. The algebra is generated by the null asymptotic of electromagnetic field and the time asymptotic of charged matter fields which incorporate the corresponding Coulomb fields. As a consequence Gauss' law is satisfied in the algebraic setting. Within this algebra the observables can be identified by the principle of gauge invariance. A class of representations of the asymptotic algebra is constructed which resembles the Kulish-Faddeev treatment of electrically charged asymptotic fields.

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*Alexander von Humboldt Fellow; present address:
Institute of Physics, Jagellonian University, Reymonta 4, 30-059 Cracow, Poland;
e-mail: herdegen@thrisc.if.uj.edu.pl

1 Introduction

It is frequently stated that the excellent experimental confirmation of quantum electrodynamics is not matched by sufficient understanding of its theoretical foundations yet. Indeed, the need for better understanding of the theory seems to be confirmed by steady fundamental research. One could even point out that experimental verification has to be considered as provisory, as long as the theory has no completely firm status. This refers not only to the existing experimental evidence. New experimental arrangements may be needed to test the results of further theoretical investigations.

This may especially be the case in relation to the problems connected with the long-range character of the electromagnetic interaction, Gauss' law and the proper description of charged states. These problems manifest themselves in the infrared divergencies of perturbational QED [1], the structure of uncountably many superselection sectors [2, 3], the infraparticle problem [4, 5] and the spontaneous breaking of the Lorentz group in charged representations of local observables [6, 5]. All these questions have been investigated in various theoretical set-ups, with varying emphasis on mathematical rigour on the one hand, and concrete calculations on the other. An important step in the developement of understanding of the long-range structure was the realization, that it is the timelike resp. lightlike asymptotic structure that is relevant here [7, 8, 9, 2]. The work of Kulish and Faddeev deserves special mentioning, as it generalized Dollard's idea [10] of asymptotic dynamics to the Gupta-Bleuler formulation of quantum electrodynamics. A more careful analysis of asymptotical charged states within this formalism was given by Morchio and Strocchi [11]. In rigorous mathematical terms, within the algebraic framework in quantum field theory, the asymptotic electromagnetic field has been obtained as an LSZ-type limit by Buchholz [3]. For reviews see the book by Jauch and Rohrlich [1], the lecture notes by Morchio and Strocchi [12], and the book by Haag [13].

In spite of the progress brought by these works, our understanding of what an electron is, is still not very concrete. So far there do only exist abstract, though rigorous, characterizations of electrically charged particles [14, 15]. Even more importantly, the concrete algebraic structure of the asymptotic quantum fields is still unclear. Thus investigations of properties such as the spontaneous breaking of the Lorentz group in charged sectors or the nonexistence of sharp masses (infraparticles) are frequently based on ad hoc assumptions. Most general results are therefore of the "no-go" type (see e.g. [5]).

It is the aim of the present work to propose a concrete formulation of the algebraic structure of asymptotic fields (and observables) of electrons interacting with radiation. This algebra resembles various elements of standard knowledge on the infrared problem. However, we would like to point out three novel aspects of our formulation.

(i) A clear and consistent algebraic framework is obtained. The asymptotic algebra is a C^*-algebra, thereby placing the problem on firm mathematical grounds.
(ii) We do not have to consider the fully interacting quantum field theory. In particular, the canonical quantization (which is taken for granted in the treatment of Kulish and Faddeev) is avoided. In fact, this heuristic procedure for obtaining the algebraic structure may fail in the field of long-range problems, as we shall discuss. Instead we base our construction on the asymptotic structure of classical electrodynamics (Maxwell and Dirac) which has been established in [16]. These results lead naturally via the correspondence principle to our quantum algebra.
(iii) The algebra incorporates the Coulomb fields of the asymptotic (outgoing) particles. In this respect it resembles the theory of the quantized Coulomb field of Staruszkiewicz [17]. However, the latter is an idealized theory of certain isolated degrees of freedom, and it seems to have no natural embedding into a larger scheme.

The plan of the paper is as follows. In Sec.2 we describe briefly the asymptotic structure of the classical theory. Part of the material is shifted to the Appendix. In Sec.3 this structure is then quantized according to the correspondence principle. Heuristical physical considerations are presented which lead to formal algebraic relations for the fields. These relations are made mathematically precise in Sec.4, and lead to our asymptotic C^*-algebra. A class of representations of the algebra is constructed in Sec.5. The question of physical relevance of these representations is left for future work. Sec.6 brings some final remarks, and comments on future perspectives of this work.

2 Classical asymptotic structure

In this section we give a short account of the asymptotic structure of a classical field theory with electromagnetic interaction, discussed at length in [16]. For the definiteness we considered the Dirac field interacting with radiation. Complete rigorous results were obtained along the lines summarized here for the both external field problems, but the extension to the full theory is possible, as argued in [16], under plausible conjectures. (The relation of our asymptotic variables to those used by Flato et al. [18] in their recent solution of the Cauchy problem and proof of asymptotic completeness of the Maxwell-Dirac system is an open problem.) The aim of this summary is also, for the convenience of the reader, to rewrite in the tensor form the properties and formulas for the asymptotics of the electromagnetic field which were discussed in the two-component spinor language in [16]. Equivalence to the original formulation is proved in the Appendix.

The idea of the approach deviates from the standard formulation of the

scattering problem as the limit of constant time configurations with time tending to infinity. Rather, the advantage is taken of the different propagation velocities of matter and radiation, to consider their asymptotics in different spacetime regions. For the electromagnetic field the known methods of the null infinity asymptotics [19, 20] are applied, formulated in terms of homogeneous functions (without the Penrose's spacetime compactification) and further developed in some specific aspects. For the matter field a method is developed which leads to the determination of an asymptotic field inside the forward lightcone – this is sufficient, as eventually every massive particle enters the cone. Plausible arguments then indicate that asymptotics thus defined contain the full information on the system and the total Poincaré quantities (energy-momentum and four-dimensional angular momentum) of the theory may be expressed in terms of them.

Electromagnetic field of the system admits a class of gauges of the potential with the null asymptotic of the form (Eq.(2.45) in [16])

$$\lim_{R\to\infty} R\,A_a(x+Rl) = V_a(x\cdot l, l)\,. \tag{2.1}$$

Here x is any spacetime point in Minkowski space M and l is a future-pointing null vector. The function $V_a(s,l)$ (s is a real variable) is homogeneous of degree -1

$$V_a(\mu s, \mu l) = \mu^{-1} V_a(s,l)\,, \tag{2.2}$$

($\mu > 0$), and satisfies

$$l\cdot V(s,l) = Q\,, \tag{2.3}$$

where Q is the charge of the field. Its s-derivative $\dot{V}_a(s,l) := \dfrac{\partial}{\partial s} V_a(s,l)$ falls off according to

$$|\dot{V}_a(s,l)| < \frac{\text{const.}}{(1+|s|)^{1+\epsilon}} \tag{2.4}$$

for some $\epsilon > 0$, so $V_a(s,l)$ has limits for $s \to \pm\infty$, which we denote $V_a(\pm\infty, l)$. (Null vectors l are scaled in (2.4) to $l^0 = 1$ in arbitrarily chosen, fixed Minkowski frame; change of frame results only in change of bounding constant.) Gauge freedom consists of the transformation $V_a(s,l) \to V_a(s,l) + \alpha(s,l) l_a$.

Further properties of the limit values $V_a(\pm\infty, l)$ involve differentiation on cone variables. A simple and explicitely Lorentz-covariant way to express the differentiations in directions tangent to the cone is to use the operator $l_a\partial_b - l_b\partial_a$ ($\partial_a := \partial/\partial l^a$). One applies the operator to any differentiable extension to some neighbourhood of the cone of a function defined on the cone itself, and restricts the result again to the cone. The result is independent of the extension used.

The limit values $V_a(\pm\infty, l)$ are constrained by the following gauge-invariant condition

$$l_{[a}\partial_b V_{c]}(\pm\infty, l) = 0 \tag{2.5}$$

(this is the tensor form of the conditions (3.32) with (2.54) – (2.57) in [16]; see also the Appendix). The physical content of the condition, which is satisfied in standard scattering situations, is, that it allows the identification of the total angular momentum of the system, as discussed in [16]. The properties (2.2), (2.3) and (2.5) allow another simple representation of $V_a(\pm\infty, l)$. It follows from (2.2) and (2.3) that $(l_b\partial_a - l_a\partial_b)V^b(\pm\infty, l) + V_a(\pm\infty, l) \propto l_a$. We denote

$$(l_b\partial_a - l_a\partial_b)V^b(+\infty, l) + V_a(+\infty, l) = 2q(l)l_a\,, \tag{2.6}$$

$$(l_b\partial_a - l_a\partial_b)V^b(-\infty, l) + V_a(-\infty, l) = 2\kappa(l)l_a\,. \tag{2.7}$$

$q(l)$ and $\kappa(l)$ are homogeneous functions of degree -2. If, for the sake of differentiation, the extensions of $V_a(\pm\infty, l)$ are chosen so as to satisfy (2.3) also in a neighbourhood of the cone, then

$$q(l) = -\frac{1}{2}\partial_b V^b(+\infty, l) \quad \kappa(l) == -\frac{1}{2}\partial_b V^b(-\infty, l)\,. \tag{2.8}$$

The charge of the field is recovered from the functions $q(l)$ and $\kappa(l)$ by the formulas

$$Q = \frac{1}{2\pi}\int q(l)d^2l = \frac{1}{2\pi}\int \kappa(l)d^2l\,. \tag{2.9}$$

By d^2l we denote that measure on the set of null directions which gives a Lorentz invariant result when applied to a homogeneous function of l of degree -2 (see [16] and references given there). The measure itself is homogeneous of degree 2, and for l's scaled to $l^0 = 1$ in any fixed Minkowski frame it is the rotationally invariant measure on the unit sphere of vectors $\vec{l}$. For future reference we note that if $\alpha(l)$ is differentiable and homogeneous of degree -2, then (see the Appendix)

$$\int (l_a\partial_b - l_b\partial_a)\alpha(l)d^2l = 0\,. \tag{2.10}$$

Now conversely, one can show that if $q(l)$ and $\kappa(l)$ are homogeneous functions of degree -2 satisfying (2.9), then the vector functions $V_a(\pm\infty, l)$ constrained by (2.2), (2.3) and (2.5) are determined by (2.6) and (2.7) uniquely up to a gauge ([16], after Eq. (2.61)).

Physical interpretation of $\kappa(l)$ is of importance: this function determines the flux of the electromagnetic field in spacelike infinity. Explicitely, if x is any point and y a spacelike vector, then

$$\lim_{R\to\infty} R^2 F_{ab}(x + Ry) = K_a(y)y_b - K_b(y)y_a\,, \tag{2.11}$$

where

$$K_a(y) = \frac{1}{2\pi y^2}\nabla_a \int \kappa(l)\,\mathrm{sgn}\, y\cdot l\, d^2l\,. \tag{2.12}$$

If the potential is decomposed in standard way into the advanced and the free outgoing part $A_a = A^{\rm adv}{}_a + A^{\rm out}{}_a$, then $A^{\rm adv}{}_a$ and $A^{\rm out}{}_a$ have again asymptotics of the type (2.1) with $V_a(+\infty, l)$ and $V^{\rm out}{}_a(s,l) := V_a(s,l) - V_a(+\infty, l)$ replacing $V_a(s,l)$ on the right hand side (rhs) of (2.1) respectively. This brings the physical interpretation of $q(l)$: this function determines the asymptotic Coulomb field of the outgoing currents.

The free outgoing field is completely determined by its asymptotic according to the formula

$$A^{\rm out}{}_a(x) = -\frac{1}{2\pi}\int \dot{V}^{\rm out}{}_a(x\cdot l, l)\, d^2 l\,. \tag{2.13}$$

The flux of the field $F^{\rm out}{}_{ab}(x)$ at the spacelike infinity is again given by the formulas (2.11) and (2.12) in which $\kappa(l)$ is replaced by $\sigma(l) := \kappa(l) - q(l) = -\frac{1}{2}\partial^b V^{\rm out}{}_b(-\infty, l)$, which is therefore interpreted as the infrared characteristic of the free field. The vector function $V^{\rm out}{}_a(-\infty, l)$ is again determined up to a gauge by $\sigma(l)$, but in this case a more explicit representation is possible. One shows that the equation (2.5) (satisfied by $V^{\rm out}{}_a(-\infty, l)$) and the condition $l\cdot V^{\rm out}(-\infty, l) = 0$ together imply the existence of a real function $\Phi(l)$ homogeneous of degree 0 such that (see the Appendix)

$$l_{[a}V^{\rm out}{}_{b]}(-\infty, l) = l_{[a}\partial_{b]}\Phi(l)\,. \tag{2.14}$$

$\Phi(l)$ is determined by this condition up to an additive constant. We make the choice of this constant characterize classes of gauges of $V^{\rm out}{}_a(-\infty, l)$ as follows. First, fix $\Phi(l)$ with some choice of the constant. Then, choose an antisymmetric real tensor function $G_{ab}(l) = -G_{ba}(l)$, homogeneous of degree 0, such that

$$l_{[a}G_{bc]}(l) = 0\,, \quad G_{ab}(l)\, l^b = \Phi(l)\, l_a\,. \tag{2.15}$$

This tensor is then of the form $G_{ab}(l) = l_a g_b(l) - l_b g_a(l)$, where $g_a(l)$ is homogeneous of degree -1, satisfies

$$l\cdot g(l) = \Phi(l)\,, \tag{2.16}$$

and is determined by $G_{ab}(l)$ up to $g_a(l) \to g_a(l) + \alpha(l) l_a$. Finally, put

$$V^{\rm out}{}_a(-\infty, l) = (l_b\partial_a - l_a\partial_b)g^b(l) + g_a(l)\,. \tag{2.17}$$

One shows that
(i) (2.17) satisfies (2.14);
(ii) every gauge of $V^{\rm out}{}_a(-\infty, l)$ may be represented in this way and the correspondence with the tensors $G_{ab}(l)$ is $1:1$;
(iii) gauges corresponding to the choices of G_{ab}'s with a fixed constant in $\Phi(l)$ form equivalence classes with respect to the following equivalence relation: $V_1^{\rm out}{}_a(-\infty, l) \sim V_2^{\rm out}{}_a(-\infty, l)$ iff $\int \alpha(l)\, d^2 l = 0$, where

${V_1^{\mathrm{out}}}_a(-\infty,l) - {V_2^{\mathrm{out}}}_a(-\infty,l) = \alpha(l)l_a$. We shall see in the next section that the interpretation imposed by the above procedure on the additive constant in $\Phi(l)$ is a very natural one from the point of view of the symplectic form for electromagnetic field.

The asymptotic of the Dirac field $\psi(x)$ inside the forward lightcone is determined by considering the behaviour of $\psi(\lambda v)$ for large λ, where the four-velocity v lies on the hyperboloid $H := \{v \in M|\, v^2 = 1, v^0 > 0\}$. Physically, the most important condition for the validity of our discussion is such a choice of gauge of the electromagnetic potential, that $|v\cdot A(\lambda v)| <$ const.$\lambda^{-1-\epsilon}$, $(\epsilon > 0)$. That this is possible in our context is shown in [16], where further details are also given. One shows then, that if we put

$$\psi(\lambda v) = -i\,\lambda^{-3/2}\, e^{-i(m\lambda + \pi/4)\gamma\cdot v} f_\lambda(v)$$

(γ^a are the Dirac matrices), then $\lim_{\lambda\to\infty} f_\lambda(v) = f(v)$. For the external field problem it is shown that this limit is reached strongly in the Hilbert space $\mathcal{K}$ of four-component functions on the hyperboloid H with the scalar product

$$(f,g) = \int \overline{f(v)}\,\gamma\cdot v\, g(v)\, d\mu(v)\,, \tag{2.18}$$

where bar denotes the usual Dirac conjugation and $d\mu(v) = d^3v/v^0$ is the invariant measure on the hyperboloid. Moreover, free outgoing field may be determined by

$$\psi^{\mathrm{out}}(x) = \left(\frac{m}{2\pi}\right)^{3/2} \int e^{-im\, x\cdot v\,\gamma\cdot v} \gamma\cdot v\, f(v)\, d\mu(v)\,. \tag{2.19}$$

$\psi^{\mathrm{out}}(x)$ has the same asymptotic as $\psi(x)$. It is argued that the structure is essentially the same in the full theory. In that case the characteristic $q(l)$ of the Coulomb field of the outgoing currents is expressed in terms of their asymptotic

$$q(l) = e \int \overline{f(v)}\,\gamma\cdot v\, f(v)\, \frac{d\mu(v)}{2(v\cdot l)^2}\,, \tag{2.20}$$

(e is the charge of the electron).

We note for future use that explicitely Lorentz-covariant differentiation on H may be discussed with the use of operators defined in exactly the same way as it has been done in the case of the lightcone in the paragraph preceding (2.5). In the case of hyperboloid the operator contracted with v^b contains the whole information

$$\delta_a f(v) := (\partial_a - v_a v\cdot\partial) f(v)\,. \tag{2.21}$$

Finally, total Poincaré quantities are expressed in terms of the outgoing fields. One finds that these quantities are the sums of the respective quantities for the free fields ${F^{\mathrm{out}}}_{ab}(x)$ and $\psi^{\mathrm{out}}(x)$ (deteremined by (2.13)

and (2.19) respectively), with, however, one additional term in the case of angular momentum

$$\Delta M_{ab} = -\frac{1}{2\pi}\int q(l)(l_a\partial_b - l_b\partial_a)\Phi(l)\, d^2l\,. \tag{2.22}$$

This term is seen to mix the outgoing matter characteristic $q(l)$ with the infrared characteristic $\Phi(l)$. (It originates from the mixed adv-out terms of the asymptotics in null directions of the electromagnetic energy-momentum tensor.) This mixing of the long-range degrees of freedom corresponds to this remnant of interaction which is responsible for the validity of the Gauss' law. Its appearence shows, that a Poisson bracket structure separating the two fields asymptotically remains in contradiction not only with the Gauss' law, but also with the Poincaré structure of the theory.

3 Quantization

We assume now that the asymptotic structure of the quantum theory may be described by quantum variables analogous to $V_a(s,l)$ and $f(v)$. By that we mean, that these analogs generate an algebra, the states of which may be interpreted as scattering states in quantum electrodynamics. In the present section we give heuristic arguments which lead us to the formulation of quantization conditions for these variables. In the next section then the appropriate algebra is constructed.

The usual quantization of the free electromagnetic field is achieved by the use of the symplectic form

$$\{F_1, F_2\} = \frac{1}{4\pi}\int_\Sigma ({F_1}^{ab}A_{2b} - {F_2}^{ab}A_{1b})\, d\sigma_a\,, \tag{3.1}$$

the integration extending over a Cauchy surface Σ. It has been observed by other authors before [19, 20] that the integration surface may be shifted so as to become the future null infinity hypersurface (in the language of the compactified Minkowski space), as the fields are determined by their data on this surface. This corresponds in our language (no compactification) to the integration of "radiated" symplectic form. This is calculated in exactly the same way in which the radiated energy-momentum and angular momentum were determined in [16], Eq.(3.6–3.14). The explicitely Lorentz-invariant result, denoted by $\{V_1, V_2\}$, is

$$\{V_1, V_2\} = \frac{1}{4\pi}\int (\dot V_1\cdot V_2 - \dot V_2\cdot V_1)(s,l)\, ds\, d^2l\,, \tag{3.2}$$

where V_i are asymptotics (2.1). However, we observe that also in the presence of sources the electromagnetic field is locally free in the null asymptotic region, so one can try to use the same symplectic form for the asymptotics

of the interacting theory. The form (3.2) is now extended without formal change to asymptotics of all fields admitted by the framework of Sec.2. (The reason for taking (3.2) rather then directly the "radiated" analog of (3.1) for general fields as a basis for generalization is that for charged fields with nonvanishing infrared part the latter form yields no Lorentz invariant result. If the calculation is performed in a frame with the time-axis along the positive unit timelike vector t then the result differs from (3.2) by

$$\frac{Q_2}{4\pi}\int t^a V_1^{\mathrm{out}}{}_a(-\infty,l)\,\frac{d^2l}{t{\cdot}l}-\frac{Q_1}{4\pi}\int t^a V_2^{\mathrm{out}}{}_a(-\infty,l)\,\frac{d^2l}{t{\cdot}l}\,,$$

where Q_i are the charges (2.3).)

If V's are split into the free and the Coulomb part $V_a(s,l)=V^{\mathrm{out}}{}_a(s,l)+V_a(+\infty,l)$ then

$$\begin{aligned}&\{V_1,V_2\}=\{V_1^{\mathrm{out}},V_2^{\mathrm{out}}\}\\&+\frac{1}{4\pi}\int\big(V_1(+\infty,l){\cdot}V_2^{\mathrm{out}}(-\infty,l)-V_2(+\infty,l){\cdot}V_1^{\mathrm{out}}(-\infty,l)\big)\,d^2l\,.\end{aligned}$$

Substituting in the second term (2.17), integrating by parts with the use of (2.10), and finally using (2.6) and (2.16) one obtains

$$\begin{aligned}\{V_1,V_2\}=&\frac{1}{4\pi}\int(\dot V_1^{\mathrm{out}}{\cdot}V_2^{\mathrm{out}}-\dot V_2^{\mathrm{out}}{\cdot}V_1^{\mathrm{out}})(s,l)\,ds\,d^2l\\&+\frac{1}{2\pi}\int(q_1\Phi_2-q_2\Phi_1)(l)\,d^2l\,.\end{aligned}\tag{3.3}$$

The first term on the rhs is gauge invariant, while the second depends on gauge only through the choice of the additive constant in $\Phi(l)$, that is on the choice of one of the equivalence classes of $V^{\mathrm{out}}{}_a(-\infty,l)$ discussed after Eq. (2.17). The above compact form of the second term supplies justification for our interpretation of the constant in $\Phi(l)$.

Let now $V^{\mathrm{op}}{}_a(s,l)$ be a quantum field and $V_a(s,l)$ a classical test field. The heuristic quantization rule is

$$[\{V_1,V^{\mathrm{op}}\},\{V_2,V^{\mathrm{op}}\}]=i\{V_1,V_2\}\,,$$

where the real multiplicative constant on the rhs is fixed by the condition, that the quantization reduces to the standard one for free infrared-regular test fields. In the Weyl exponentiated form this becomes $W(V_1)W(V_2)=e^{-(i\beta^2/2)\{V_1,V_2\}}W(V_1+V_2)$, where we put $W(V)=e^{-i\beta\{V,V^{\mathrm{op}}\}}$, with β a real constant to be determined shortly. The Weyl operators are assumed to depend only on those variables, which enter nontrivially into the symplectic form (3.3), that is they are insensitive to the gauge of $V^{\mathrm{out}}{}_a(s,l)$ for finite s and to the gauge of $V_a(+\infty,l)$, and they depend on the gauge of $V^{\mathrm{out}}{}_a(-\infty,l)$ only through the choice of constant in $\Phi(l)$. Therefore we shall write sometimes $W(V)=W(\xi,\Phi,q)$, where $\xi_{ab}(s,l):=l_aV^{\mathrm{out}}{}_b(s,l)-l_bV^{\mathrm{out}}{}_a(s,l)$. (Remember that $\Phi(l)$ is determined by $\xi_{ab}(-\infty,l)$ up to an additive constant.)

The form (3.3) determines the physical interpretation of the Weyl operator for $q = 0$, $\xi = 0$, $\Phi = c = \text{const.}$: $W(0, c, 0) = e^{i\beta c Q^{\mathrm{op}}}$, where Q^{op} is the operator of the charge of the field. From Weyl relations we have then

$$e^{i\beta c Q^{\mathrm{op}}} W(V) = W(V) e^{i\beta c (Q^{\mathrm{op}} + \beta Q)} ,$$

where Q is the charge of the electromagnetic test field V. For the interpretation of the classical and the quantum charge to agree we set $\beta = 1$. Then the Weyl operator $W(V)$ carries a quantum charge equal to the classical charge of the test field V. More generally, it may be interpreted to carry the asymptotic field characterized by $V_a(s, l)$. The Weyl algebra

$$W(V_1) W(V_2) = e^{-(i/2)\{V_1, V_2\}} W(V_1 + V_2) \tag{3.4}$$

may be considered as a theory of the asymptotic electromagnetic field. The quantization of charge demands that the space of test fields be restricted to the abelian additive group of those V's which carry the multiple of the elementary charge $Q = (1/2\pi) \int q(l) d^2 l = ne$. The subgroup of zero-charge test fields forms a vector space. (For the discussion of an "adiabatic limit" of such a theory, in which only the long-range characteristics of the field survive, we refer the reader to [21].) However, the theory thus formulated is physically incomplete – it admits Coulomb fields, but there are no particles present to carry these fields. We turn now to the description of these particles.

Let us forget for the moment that there is some "Gauss coupling" between the asymptotic electromagnetic and Dirac fields which has to modify the Poisson bracket structure (as compared with the structure of two independent fields). Then the quantum field $f^{\mathrm{op}}(v)$ which is to correspond to the classical $f(v)$ is quantized in the standard Dirac way. We denote the quantum field smeared with the test field $f(v)$ by $B(f)$ and replace $f^{\mathrm{op}}(v)$ by $B(v)$, so that symbolically $B(f) = \int \overline{f(v)}\, \gamma\cdot v\, B(v)\, d\mu(v)$, where f is in $\mathcal{K}$, the Hilbert space introduced before (2.18). The standard quantization law in our notation reads

$$[B(f), B(g)]_+ = 0 , \quad [B(f), B(g)^*]_+ = (f, g) .$$

It will be convenient for our purposes to have fermionic operators depending on f's linearly. We introduce notation

$$B^0(f) = B(f^c) = \int B^0(v)\, \gamma\cdot v\, f(v)\, d\mu(v) , \tag{3.5}$$

$$\bar{B}(f) = B(f)^* = \int \bar{B}(v)\, \gamma\cdot v\, f(v)\, d\mu(v) , \tag{3.6}$$

where f^c is the charge conjugation of f defined by

$$f^c = C\overline{f}^{\mathrm{T}} , \tag{3.7}$$

with C a unitary, antisymmetric matrix inducing the transformation $C^{-1}\gamma^a C = -\gamma^{a\,\mathrm{T}}$. The involution law is then

$$B^0(f)^* = \bar{B}(f^c)\,, \tag{3.8}$$

and anticommutation relations

$$[B^0(f), B^0(g)]_+ = 0\,, \quad [B^0(f), \bar{B}(g)]_+ = (f^c, g)\,, \tag{3.9}$$

or, symbolically,

$$[B^0(v)_\alpha, B^0(v)_\beta]_+ = 0\,, \quad [B^0(v)_\alpha, \bar{B}(u)_\beta]_+ = \delta(v,u)(C^{-1}\gamma\cdot v)_{\alpha\beta}\,, \tag{3.10}$$

where $\delta(v,u)$ is the Dirac "δ-function" in the two velocities with respect to the measure $d\mu(v)$.

Physical interpretation of $B(f)$ and $B^0(f)$ justified by the Fock representation is, that these operators annihilate an electron and/or create a positron. This means that they locally create charge $-e$ (if e is the charge of electron). More specifically, the operators $B(v)$ and $B^0(v)$ (resp. $\bar{B}(v)$) (forget for the moment mathematical subtleties) create the charge $-e$ (resp. e) moving with a constant four-velocity v. However, if the Gauss' law is again brought into play, creation of a charged particle must have electromagnetic consequences. Therefore, we want to extend the effect of B's in such a way, that they create (or annihilate) also the Coulomb field accompanying the charge. Basing the intuitions on pictures from perturbation calculations and on the algebraic discussion of superselection sectors of local observables, we want to admit the possibility, that charged particles are in addition accompanied by "clouds" of radiation. Let the potential of the total field (Coulomb + radiation) accompaning charge e moving with velocity v be characterized by the asymptotic $V_a(v) = V_a(v; s, l)$. Then for each v this asymptotic is in the class discussed in Sec.2, and, moreover, its Coulomb part is, up to a gauge, the asymptotic of the potential $A_a(x) = ev_a/v\cdot x$, that is $V_a(v; +\infty, l) = ev_a/v\cdot l +$ gauge. Now we seek operators $B^0_{-V_1}(v)$ and $\bar{B}_{V_2}(v)$ analogous to $B^0(v)$ and $\bar{B}(v)$ respectively, which, however, beside creating or annihilating material particles should also carry accompaning electromagnetic fields with asymptotics $-V_1(v)$ and $V_2(v)$ respectively, where V_1 and V_2 are in the class introduced above. Thus formulated, the problem almost uniquely determines its solution – the objects which do the electromagnetic part of the task are already there, the Weyl operators $W(-V_1(v))$ and $W(V_2(v))$ carry exactly those charged fields. For the purpose of obtaining commutation relations we imagine the operators $B^0_{-V_1}(v)$ (resp. $\bar{B}_{V_2}(v)$) to be formed as products of $B^0(v)$ (resp. $\bar{B}(v)$) and $W(-V_1(v))$ (resp. $W(V_2(v))$) (mutually commuting). On the other hand, having attached charged fields to the matter particles in this way, we do not need any longer, as independent objects, the Weyl operators $W(V)$ for test fields with nonvanishing Coulomb part. From now on the asymptotic V in $W(V)$ is always a free field asymptotic ($V_a(+\infty, l) = 0$).

The (naive) commutation relations resulting from the above discussion are

$$B^{0}_{-V_1}(v)W(V) = e^{(i/2)\{V_1(v),V\}}B^{0}_{-V_1+V}(v)\,,$$

$$\bar{B}_{V_2}(v)W(V) = e^{-(i/2)\{V_2(v),V\}}\bar{B}_{V_2+V}(v)\,,$$

$$e^{(i/2)\{V_1(v),V_2(u)\}}B^{0}_{-V_1}(v)_\alpha B^{0}_{-V_2}(u)_\beta + e^{(i/2)\{V_2(u),V_1(v)\}}B^{0}_{-V_2}(u)_\beta B^{0}_{-V_1}(v)_\alpha = 0\,,$$

$$e^{-(i/2)\{V_1(v),V_2(u)\}}B^{0}_{-V_1}(v)_\alpha \bar{B}_{V_2}(u)_\beta + e^{-(i/2)\{V_2(u),V_1(v)\}}\bar{B}_{V_2}(u)_\beta B^{0}_{-V_1}(v)_\alpha$$

$$= \delta(v,u)(C^{-1}\gamma\cdot v)_{\alpha\beta}W(V_2(v)-V_1(v))\,,$$

supplemented with the Weyl relations. Note that the asymptotic in the Weyl operator in the last line is a free field asymptotic, as $V_2(v;+\infty,l) - V_1(v;+\infty,l) = 0$.

For a precise formulation of the above quantization conditions it will not suffice, in contrast to the case of the Dirac field algebra, to have objects smeared with one-particle test functions as generating elements. Smeared products of $B^{0}_{-V_1}(v)$, $\bar{B}_{V_2}(v)$ and $W(V_2(v)-V_1(v))$ have to be defined, and the phase factors appearing in the above relations must become multipliers in the space of test fields. This construction is given in the next section.

Before we go over to this task we want to draw attention to a known fact concerning physical interpretation of Weyl operators, which, however, in our context is of decisive importance and must not be overlooked. We illustrate our point first on the simplest possible example, the Weyl algebra of a single pair of canonical variables in quantum mechanics, $[x,p] = i\mathbf{1}$. The Weyl formulation reads in that case

$$W(x_1,p_1)W(x_2,p_2) = e^{-(i/2)\{x_1,p_1;x_2,p_2\}}W(x_1+x_2,p_1+p_2)\,,$$

where x_i and p_i are classical "test"- position and momentum variables, the symplectic form is $\{x_1,p_1;x_2,p_2\} = x_1p_2 - x_2p_1$, and the Weyl operator is interpreted as $W(x_1,p_1) = e^{-i\{x_1,p_1;x,p\}}$. We want to point out a certain duality in the interpretation of $W(x_1,p_1)$. Let us set for simplicity $p_1 = 0$. Then, on the one hand, if treated as a function of observables, the Weyl operator $W(x_1,0) = e^{-ix_1p}$ "measures" the probability distribution of a state with respect to the momentum. On the other hand we have $W(x_1,0)^*x\,W(x_1,0) = x + x_1$. Hence, if treated as a unitary transformation operator, $W(x_1,0)$ "carries" the translation x_1, that is when applied to a vector in Hilbert space it increases its position characteristic by the "test"-position x_1. The characteristic which is "carried" by a Weyl operator in this sense is thus the one given by the test-quantity, while the "measured" one is dual to it, in the sense of the symplectic form. Going back to our objects we see that $B^{0}_{-V_1}(v)$, $\bar{B}_{V_2}(v)$ and $W(V)$ carry the respective fields in the above sense, in the case of $W(V)$ a free field. However, from the symplectic form

(3.3) one reads off, that the asymptotic of a free field potential has as its dual the asymptotic of the total field, which is therefore what is "measured" by $W(V)$.

4 The algebra

Consider the set of $\mathcal{C}^\infty$ functions $V_a(s,l)$ (differentiations with respect to l in the sense discussed before Eq.(2.5), outside some neighbourhood of the vertex of the cone) satisfying conditions (2.2 – 2.5). In this set introduce the following equivalence relation: $V_2(s,l) \sim V_1(s,l)$ iff $V_2(s,l) = V_1(s,l) + \alpha(s,l)\, l$ and $\int (\alpha(-\infty,l) - \alpha(+\infty,l))\, d^2l = 0$ (this is the equivalence relation for the infrared characteristics of the free field component of V_a, introduced in (iii) after (2.17)). The set of equivalence classes with respect to this relation will be denoted by L_Q. Another way of characterizing elements of L_Q is by the triples (ξ, Φ, q) introduced in the paragraph preceding Eq.(3.4). In order not to burden the notation, the elements of L_Q will be denoted by $V_a(s,l)$, but always the equivalence classes are understood. The set $\hat{L} := \bigcup_{n\in\mathbf{Z}} L_{ne}$ has in natural way the structure of an abelian additive group. With the map $\{.,.\} : \hat{L} \times \hat{L} \to \mathbf{R}$ defined by (3.2) it becomes a symplectic group, on which $\{.,.\}$ is nondegenerate. The subgroup $L_0 \subset \hat{L}$ has the structure of a vector space. Its subspace consisting of elements satisfying in addition $V_a(+\infty,l) \propto l_a$ (no Coulomb field) will be denoted by L. Without loss of generality it may be assumed that for all V's in L there is $V_a(+\infty,l) = 0$. Elements of L will be the test functions of Weyl operators.

Consider, next, the class of all functions $V_a(v)$ on the hyperboloid H with values in L_e, such that

(i) $V^a(v;+\infty,l) = {V_e}^a(v,l) + \text{gauge}$, where ${V_e}^a(v,l) := ev^a/v\cdot l$, or, equivalently, $q(v,l) = q_e(v,l) := (e/2)(v\cdot l)^{-2}$;

(ii) for each $V(v)$ there is a function $F_V(v)$ such that

$$\{V(v), V(u)\} = F_V(v) - F_V(u) \tag{4.1}$$

for every v and u;

(iii) representants $V_a(v;s,l)$ may be chosen in $\mathcal{C}^\infty$ in v (differentiations in the sense of (2.21)) and for each $k = 0,1,\ldots$ there are constants $C_k \in \mathbf{R}$ and $m_k \in \mathbf{N} \cup \{0\}$ such that

$$\left|\delta_{a_1} \ldots \delta_{a_k} \dot{V}_b(v;s,l)\right| < C_k (v^0)^{m_k} (|s|+1)^{-1-\epsilon}$$

(in a fixed Minkowski frame, with scaling of l's fixed by $l^0 = 1$; the change of frame induces only a change of C_k).

Let $\mathcal{S}at_e$ be a subclass of this family of functions, such that

(iv) if $V_1(v) \in \mathcal{S}at_e$ and $V_0 \in L$ then $V_2(v) = V_1(v) + V_0 \in \mathcal{S}at_e$; this condition is fulfilled if $\mathcal{S}at_e$ consists of all functions satisfying (i)–(iii).

Denote, moreover, $\mathcal{S}at_{-e} := -\mathcal{S}at_e$ and $\mathcal{R}ad := \mathcal{S}at_e + \mathcal{S}at_{-e}$. The elements of $\mathcal{S}at_e$ and $\mathcal{S}at_{-e}$ will be the fields accompanying particles. Free fields from the class $\mathcal{R}ad$ will serve to define smeared Weyl operators.

Physical meaning of the first condition has been explained before. The next two conditions are of technical nature. The second one will guarantee the boundedness of the fermionic operators. The third imples that for $V_1, V_2 \in \mathcal{S}at_e \cup \mathcal{S}at_{-e} \cup \mathcal{R}ad$ the symplectic form $\{V_1(v), V_2(u)\}$, and phase factors containing it linearly in exponent, are $\mathcal{C}^\infty$ functions in both variables, bounded polynomially in each of them. These properties make them to multipliers in the space of Schwartz functions on $H^{\times n}$. Finally, the fourth condition says that a free field may be added to the cloud of the particle.

After these preliminaries our algebra may be constructed. We introduce formal symbols $W(V)$ for $V \in L$, W_V for $V \in \mathcal{R}ad$, B^0_V for $V \in \mathcal{S}at_{-e}$, and $\bar{B}_V$ for $V \in \mathcal{S}at_e$. The symbol to which a given V is attached determines the class to which it belongs, so there is no need for special notation of V's for each case separately. Let D be any finite sequence of these four symbols and χ a Schwartz function having one four-velocity argument for each of the symbols W_V, B^0_V and $\bar{B}_V$, and one index taking the values $\alpha = 1, \dots, 4$, for each of the symbols B^0_V and $\bar{B}_V$. If the sequence D contains n symbols W_V, B^0_V and $\bar{B}_V$, and m symbols B^0_V and $\bar{B}_V$, then $\chi \in \mathcal{S}(H^n, \mathbf{C}^{4^m})$. We introduce a new symbol $[D](\chi)$, linear by assumption in χ. For a symbol consisting of only one of the operators W_V, B^0_V or $\bar{B}_V$ the symbolic notation is introduced

$$W_V(\chi) = \int W_V(v)\chi(v)\, d\mu(v)\,, \tag{4.2}$$

$$B^0_V(f) = \int B^0_V(v)\, \gamma{\cdot}v\, f(v)\, d\mu(v)\,, \tag{4.3}$$

$$\bar{B}_V(f) = \int \bar{B}_V(v)\, \gamma{\cdot}v\, f(v)\, d\mu(v)\,, \tag{4.4}$$

(cf. (3.5) and (3.6)) and extended by linearity to general symbols $[D](\chi)$. The set of all formal finite sums of these symbols forms a vector space. We divide this space by its subspace generated by the following identifications ($G_V(v)$ is any of the symbols $W_V(v)$, $B^0_V(v)$ or $\bar{B}_V(v)$)

$$e^{(i/2)\{V_1,V_2\}} W(V_1) W(V_2) = W(V_1 + V_2)\,, \tag{4.5}$$

$$e^{(i/2)\{V_1,V_2(v)\}} W(V_1) G_{V_2}(v) = G_{V_1+V_2}(v)\,, \tag{4.6}$$

$$e^{(i/2)\{V_2(v),V_1\}} G_{V_2}(v) W(V_1) = G_{V_1+V_2}(v)\,, \tag{4.7}$$

$$\begin{aligned} &e^{(i/2)\{V_1(v),V_2(u)\}} W_{V_1}(v) G_{V_2}(u) \\ &\qquad - e^{(i/2)\{V_2(u),V_1(v)\}} G_{V_2}(u) W_{V_1}(v) = 0\,, \end{aligned} \tag{4.8}$$

$$
\begin{aligned}
&e^{(i/2)\{V_1(v),V_2(u)\}}B^\sharp_{V_1}(v)_\alpha B^\sharp_{V_2}(u)_\beta \\
&\qquad\qquad\qquad +e^{(i/2)\{V_2(u),V_1(v)\}}B^\sharp_{V_2}(u)_\beta B^\sharp_{V_1}(v)_\alpha = 0\,,
\end{aligned}
\tag{4.9}
$$

$$
\begin{aligned}
&e^{(i/2)\{V_1(v),V_2(u)\}}B^0_{V_1}(v)_\alpha \bar{B}_{V_2}(u)_\beta \\
&\qquad\qquad\qquad +e^{(i/2)\{V_2(u),V_1(v)\}}\bar{B}_{V_2}(u)_\beta B^0_{V_1}(v)_\alpha \\
&\qquad = (\underset{\cdot}{v},u)\,(C^{-1}\gamma\cdot v)_{\alpha\beta}W_{V_1+V_2}(v)\,;
\end{aligned}
\tag{4.10}
$$

if $V(v) = V_0 = \text{const.}(v)$ on the support of $\chi(\ldots,v,\ldots)$ in v, the variable connected with $W_V(v)$, then

$$
[\ldots W_V\ldots](\chi) = [\ldots W(V_0)\ldots]\left(\int \chi(\ldots,v,\ldots)\,d\mu(v)\right)\,. \tag{4.11}
$$

In Eq.(4.9) $\sharp = {\scriptstyle 0}$ or the bar sign, the same at both B's. The phase factors appearing in (4.5 – 4.10) are to be understood to multiply test functions χ in the symbols $[D](\chi)$. The last relation says, that constant smeared Weyl operators are identical with the standard ones.

The elements of the factor space thus obtained will be again denoted by $\sum_{i=1}^N [D_i](\chi_i)$ without a risk of confusion. This vector space becomes a $*$-algebra $\mathcal{B}$ with the multiplication- and involution- law defined by

$$
[D_1](\chi_1)[D_2](\chi_2) = [D_1D_2](\chi_1\otimes\chi_2)\,, \tag{4.12}
$$

$$
[D](\chi)^* = [D^*](\chi^c)\,, \tag{4.13}
$$

and the unit $\mathbf{1} = W(0)$. Here D_1D_2 is the sequence of symbols formed of the two sequences D_1 and D_2, D_2 following D_1. χ^c results from χ by the application of the sequence of three operations: complex conjugation, reflection of the order of the variables and indices, and the matrix multiplication by $C\gamma^{0\mathrm{T}} = -\gamma^0 C$ applied to each of the indices (cf. (3.7)). The sequence D^* results from D by reflection of its order and subsequent replacements: $W(V)\to W(-V)$, $W_V\to W_{-V}$, $B^0_V\to\bar{B}_{-V}$, $\bar{B}_V\to B^0_{-V}$.

In the last step we consider now the problem of introducing a C^*-norm on the $*$-algebra $\mathcal{B}$. Let $\mathcal{R}$ be the class of all C^*-seminorms p on $\mathcal{B}$ such that

(i) $p([D](\chi))$ is continuous in χ in the topology of $\mathcal{S}$ for each D;

(ii)

$$
p\left(W_V(\chi)\right) \leq \|\chi\|_{L^1(H,d\mu)}\,. \tag{4.14}
$$

A comment on each of the conditions is in place. The second one is a necessary condition for the admitted representations of the smeared Weyl operators $W_V(\chi)$ to be indeed given by integrals of unitary operators with the test function χ. To see the meaning of the first condition let us compare our present context with that of the standard algebra of the Dirac field. In the latter case one has in the algebra the elements $[B^{\sharp_1}\ldots B^{\sharp_n}](f^1\otimes\ldots\otimes f^n) := B^{\sharp_1}(f^1)\ldots B^{\sharp_n}(f^n)$, ($\sharp_i = {\scriptstyle 0}$ or the bar sign), where $f^i\in\mathcal{S}(H,\mathbf{C}^4)\subset\mathcal{K}$.

These elements are norm continuous in each of the functions f^i in the $\mathcal{S}$-topology, so by the nuclear theorem for Schwartz functions one obtains in the algebra the unique linear extension $[B^{\sharp_1}\dots B^{\sharp_n}](\chi)$ to the whole of $\mathcal{S}(H^n,\mathbf{C}^{4^n})$, norm continuous in χ in the topology of $\mathcal{S}$. Their analogs in our algebra are symbols $[D](\chi)$. However, we had to define them from the beginning for the whole space $\mathcal{S}$ to be able to formulate the algebraic conditions. Condition (i) will guarantee that also here they will be continuous extensions of products of the basic objects.

Proposition 4.1 *The class $\mathcal{R}$ contains the maximal element p_{max}. A C^*-seminorm p on $\mathcal{B}$ is in $\mathcal{R}$ iff $p\leq p_{max}$.*

The second statement is obviously true, if the first is proved. The proof is preceded by two lemmas.

Lemma 4.2 *For any C^*-seminorm $p\neq 0$ on $\mathcal{B}$ there is $p(W(V))=1$ for all $V\in L$, and $p\left(B^0_{V_1}(f)\right)=p\left(\bar{B}_{V_2}(f)\right)=\|f\|_{\mathcal{K}}$ for all $V_1\in\mathcal{S}at_{-e}$, $V_2\in\mathcal{S}at_e$ and $f\in\mathcal{S}$.*

Proof. The first statement is the consequence of the Weyl relations. The proof of the second one is a slightly more involved version of the analogous proof for the C^*-norm on the Dirac field algebra. Let $\{V(v),V(u)\}=F(v)-F(u)$. Then (4.9) for $V_1=V_2=V$ takes the form

$$e^{iF(v)}B^0_V(v)_\alpha B^0_V(u)_\beta+e^{iF(u)}B^0_V(u)_\beta B^0_V(v)_\alpha=0\,.$$

For $\chi=f\otimes g$ this yields $B^0_V(e^{iF}f)B^0_V(g)+B^0_V(e^{iF}g)B^0_V(f)=0$, and for $g=f$, in particular, $B^0_V(e^{iF}f)B^0_V(f)=0$. In the same way one gets from (4.10)

$$B^0_V(f)\bar{B}_{-V}(g)+\bar{B}_{-V}(e^{-iF}g)B^0_V(e^{iF}f)=(f^c,g)\,. \tag{4.15}$$

The last two equations imply

$$B^0_V(f)B^0_V(g)^*B^0_V(f)=(g,f)B^0_V(f)\,. \tag{4.16}$$

Multiplying this equation on the left by $B^0_V(f)^*$ and setting $g=f$ one gets easily $p\left(B^0_V(f)\right)=\|f\|_{\mathcal{K}}$ or 0. Assume that there is V and $g\neq 0$ such that $p\left(B^0_V(g)\right)=0$. Then from (4.16) there is $|(g,f)|p\left(B^0_V(f)\right)=0$ for all f, hence $p\left(B^0_V(f)\right)=0$ for all such f that $(g,f)\neq 0$. Each f may be represented as $f=f_1+f_2$ with $(g,f_i)\neq 0$, so $p\left(B^0_V(f)\right)=0$ for all f, which contradicts (4.15) and ends the proof of the lemma. □

Let $\chi\in\mathcal{S}(H^n,\mathbf{C}^{4^m})$, $m\leq n$. There always exists a representation $\chi=\sum_{i=1}^{\infty}f^1_i\otimes\ldots\otimes f^n_i$, where for a given k all f^k_i are either in $\mathcal{S}(H,\mathbf{C})$ or in $\mathcal{S}(H,\mathbf{C}^4)$ and the sum converges in the topology of $\mathcal{S}$ (e.g. the

N-representation [22]). There are various orders of spaces $\mathcal{S}(H, \mathbf{C})$ and $\mathcal{S}(H, \mathbf{C}^4)$ in this representation possible. Denote a fixed order by $\flat$ and the above representation with this order by $\chi = \sum_{i=1}^{\infty} f_i^1 \otimes_\flat \ldots \otimes_\flat f_i^n$. Let

$$d_\flat(\chi) := \inf_{\chi = \sum_{i=1}^{\infty} f_i^1 \otimes_\flat \ldots \otimes_\flat f_i^n} \sum_{i=1}^{\infty} \|f_i^1\|_\bullet \ldots \|f_i^n\|_\bullet \,, \tag{4.17}$$

where $\|f\|_\bullet = \|f\|_{L^1(H,d\mu)}$ if $f \in \mathcal{S}(H, \mathbf{C})$ and $\|f\|_\bullet = \|f\|_\mathcal{K}$ if $f \in \mathcal{S}(H, \mathbf{C}^4)$.

Lemma 4.3 *$d_\flat$ are norms on $\mathcal{S}(H^n, \mathbf{C}^{4^m})$, continuous in the topology of $\mathcal{S}$.*

Proof. We show first that $d_\flat$ is bounded by one of the seminorms defining the topology of $\mathcal{S}$. We assume for simplicity that $\chi \in \mathcal{S}(H^2, \mathbf{C}^4)$ and we are interested in the norm $d_\flat$ for the order of spaces $(\mathcal{S}(H, \mathbf{C}), \mathcal{S}(H, \mathbf{C}^4))$. The general case differs only by more involved notation. Choose a Minkowski frame and denote $\chi'_\alpha(v, u) = v^0(u^0)^{-1/2} \sum_{\beta=1}^{4} S^{-1}(u)_{\alpha\beta}\chi_\beta(v, u)$, where the matrix $S^{-1}(u) = (2(u^0 + 1))^{-1/2}(1 + \gamma^0\gamma \cdot u)$ induces the transformation $\overline{f}\gamma \cdot ug = (S^{-1}(u)f)^\dagger(S^{-1}(u)g)$ (the dagger denoting the matrix hermitian conjugation). Consider χ' as a function of variables $\vec{v}$ and $\vec{u}$ and expand it in the N-representation: $\chi'_\alpha(v, u) = \sum_{i=1}^{\infty} f_i(\vec{v})g_{i\alpha}(\vec{u})$, where f's and g's are multiples of products of the Hermite functions. The sum converges in $\mathcal{S}$ and $\sum_{i=1}^{\infty} \|f_i\|_{L^1(\mathbf{R}^3,d^3v)} \left(\sum_{\alpha=1}^{4} \|g_{i\alpha}\|_{L^2(\mathbf{R}^3,d^3v)}\right)$ is bounded by one of the fundamental seminorms of χ', which in turn may be bounded by one of the seminorms of χ. Now it suffices to observe that $\sum_{i=1}^{\infty}(v^0)^{-1} f_i(\vec{v})\,(u^0)^{1/2}\,(S(u)g_i(\vec{u}))_\alpha$ converges to χ in $\mathcal{S}$ and $\sum_{i=1}^{\infty} \|(v^0)^{-1} f_i(\vec{v})\|_{L^1(H,d\mu)}\, \|(u^0)^{1/2}\,(S(u)g_i(\vec{u}))\,\|_\mathcal{K} <$ const. $\sum_{i=1}^{\infty} \|f_i\|_{L^1(\mathbf{R}^3,d^3v)} \left(\sum_{\alpha=1}^{4} \|g_{i\alpha}\|_{L^2(\mathbf{R}^3,d^3v)}\right)$ to get $d_\flat(\chi) < \rho(\chi)$, where ρ is one of the fundamental seminorms. The properties of a seminorm are easily checked for $d_\flat$. That $d_\flat(\chi) = 0$ implies $\chi = 0$ is again illustrated on our special case. Let $\chi = \sum_{i=1}^{\infty} f_i \otimes g_i$, $f_i \in \mathcal{S}(H, \mathbf{C})$, $g_i \in \mathcal{S}(H, \mathbf{C}^4)$ and also choose any $h_1 \in \mathcal{S}(H, \mathbf{C})$ and $h_2 \in \mathcal{S}(H, \mathbf{C}^4)$. Then

$$\left|\int h_1(v)\overline{h_2(u)}\gamma \cdot u\chi(v, u)\, d\mu(v)\, d\mu(u)\right| \leq \sup_v |h_1(v)| \sum_{i=1}^{\infty} \|f_i\|_{L^1}\|g_i\|_\mathcal{K} \,.$$

Hence, if $d_\flat(\chi) = 0$ then the lhs vanishes for all h_1 and h_2 and $\chi = 0$. $\square$

Proof of Proposition 4.1. Let $\flat(D)$ be the order of spaces $\mathcal{S}(H,\mathbf{C})$ and $\mathcal{S}(H,\mathbf{C}^4)$ corresponding to the order of symbols W_V and $B^\sharp_V$, respectively, in the sequence D. Then by the assumption (4.14) and Lemma 4.2 one has for any $p\in\mathcal{R}$

$$p\left([D](f^1\otimes_{\flat(D)}\ldots\otimes_{\flat(D)}f^n)\right)\leq\|f^1\|_\bullet\ldots\|f^n\|_\bullet\,.$$

If $\chi=\sum_{i=1}^\infty f_i^1\otimes_{\flat(D)}\ldots\otimes_{\flat(D)}f_i^n$ then by the assumed continuity $p([D](\chi))\leq\sum_{i=1}^\infty\|f_i^1\|_\bullet\ldots\|f_i^n\|_\bullet$. Hence for any element of $\mathcal{B}$ one has

$$p\left(\sum_{k=1}^N[D_k](\chi_k)\right)\leq\sum_{k=1}^N d_{\flat(D_k)}(\chi)\,.$$

We define the seminorm $p_{\max}$ on $\mathcal{B}$ by

$$p_{\max}(A)=\sup_{p\in\mathcal{R}}p(A)\,.$$

There is $p_{\max}\left([D](\chi)\right)\leq d_{\flat(D)}(\chi)$, so $p_{\max}\in\mathcal{R}$. $\square$

The answer to the question whether $p_{\max}$ is a norm on $\mathcal{B}$ is not known yet. If it is not, one divides $\mathcal{B}$ through the ideal $\mathcal{I}$ of those elements for which $p_{\max}$ vanishes. The seminorm $p_{\max}$ induces then a C^*-norm $\|.\|$ on $\mathcal{B}/\mathcal{I}$ in the standard way. The completion of $\mathcal{B}/\mathcal{I}$ in this norm is a C^*-algebra $(\mathcal{F},\|.\|)$. We propose to regard this algebra as the base of a theory of asymptotic fields. With regard to the interpretation of electromagnetic ingredients of the algebra, one should have in mind the remarks made at the end of previous section.

It is easy to see that representations of the algebra $(\mathcal{F},\|.\|)$ are in natural $1:1$ correspondence with those representations π of $\mathcal{B}$ for which $p(.)=\|\pi(.)\|\in\mathcal{R}$.

In the asymptotic algebra of fields there is no place more for the local gauge transformations. The only gauge dependent quantity in the electromagnetic test fields is the additive constant in $\Phi(l)$. This freedom is closely connected with the global gauge transformation of the charge carrying fields, which is implemented in the algebra itself. Let $W(V)=W(0,c)$, $c=$ const. (i.e. $V=$ pure gauge, $\Phi=c$), and set $\gamma_c(A)=W(0,c)A\,W(0,c)^*$. Then $\gamma_c(A)=A$ for $A=W(V)$ or $W_V(\chi)$, and $\gamma_c(B^0_V(f))=e^{-ice}B^0_V(f)$,

$\gamma_c(\bar{B}_V(f))=e^{+ice}\bar{B}_V(f)$. Algebra $\mathcal{B}$ is the linear span of its subspaces $\mathcal{B}_k$, $k\in\mathbf{Z}$, where $\gamma_c=e^{icke}\,\mathrm{id}$ on $\mathcal{B}_k$. The subspace $\mathcal{B}_0$ is a $*$-subalgebra of $\mathcal{B}$. If $\mathcal{I}\neq0$ then it is easily seen that $\mathcal{I}$ is the linear span of $\mathcal{I}_k:=\mathcal{B}_k\cap\mathcal{I}$. The decomposition is therefore inherited by $\mathcal{F}$, and $\mathcal{F}_0$ is a C^*-algebra, which may be interpreted as the algebra of observables $\mathcal{A}\equiv\mathcal{F}_0$.

The algebraic relations of $\mathcal{B}$ have been obtained by treating $B^\sharp_V(v)$ heuristically as products of $B^\sharp(v)$ and Weyl operators for charged fields $W(V(v))$. In the next section we shall see that this heuristic idea may be also used to obtain a class of representations of $\mathcal{B}$ (and $\mathcal{F}$).

5 A class of representations

Let $W_0(V)$ be a representation in a Hilbert space $\mathcal{H}_1$ of the Weyl algebra over the test function space $\hat{L}$ with the symplectic form (3.2). We assume that for any $V(.) \in \mathcal{S}at_e$ it satisfies the following conditions
(i) for every $\varphi \in \mathcal{H}_1$ the vectors $W_0(V(v))\varphi$, $v \in H$, span a separable subspace;
(ii) for every φ, $\psi \in \mathcal{H}_1$ the function $(\varphi, W_0(V(v))\psi)$ is measurable in v.
(The class of representations satifying the conditions is nonempty – this may be shown by an explicit construction making use of the usual Fock representation and one of representations discussed in [21].) It follows that $(\varphi, W_0(V_1(v_1)) \dots W_0(V_n(v_n))\psi)$ is also measurable. For $V_i \in \mathcal{S}at_e \cup \mathcal{S}at_{-e} \cup \mathcal{R}ad$, $i = 1, \dots, n$, $\chi \in L^1(H^n, d\mu^n)$ we denote

$$\begin{aligned}&[W_{0V_1} \dots W_{0V_n}](\chi) = \\ &\int W_0(V_1(v_1)) \dots W_0(V_n(v_n))\chi(v_1, \dots, v_n)\, d\mu(v_1) \dots d\mu(v_n)\,,\end{aligned}$$

the integral in the weak sense. The Weyl algebra relations imply

$$[W_{0V_1} W_{0V_2}](\chi) = [W_{0V_2} W_{0V_1}](\chi')\,, \tag{5.1}$$

where $\chi'(u, v) = e^{-i\{V_1(v), V_2(u)\}}\chi(v, u)$. If, in particular, $\{V_1(v), V_2(u)\} = F_{12}(v) - F_{12}(u)$, then

$$W_{0V_1}(\chi_1) W_{0V_2}(\chi_2) = W_{0V_2}(e^{iF_{12}}\chi_2) W_{0V_1}(e^{-iF_{12}}\chi_1)\,. \tag{5.2}$$

Let further $B(f)$ (and $B^\sharp(f)$) be a concrete realization (representation) of the Dirac field algebra in a Hilbert space $\mathcal{H}_2$. We would like to define $B^0_V(v)$ as a product of $W_0(V(v))$ and $B^0(v)$, that is to give sense to the expression

$$B^0_V(f) = \int W_0(V(v)) \otimes B^0(v)\, \gamma{\cdot}v\, f(v)\, d\mu(v)$$

for $f \in \mathcal{S}(H, \mathbf{C}^4)$. (When constructing the representation we shall use the simplified notation $B^0_V(f)$ instead of the more appropriate $\pi(B^0_V(f))$ etc., which may be restored at the end of construction.) Let $\{e_i\}$ be an orthonormal basis of the Hilbert space $\mathcal{K}$. If we "expand" $B^0(v)$ in this basis, we are led to the following formulation. Consider a family of bounded operators on $\mathcal{H}_1 \otimes \mathcal{H}_2$

$$B^0_V(f)_n := \sum_{i=1}^n W_{0V}(\overline{e_i}\Gamma f) \otimes B^0(e_i)\,,$$

where $(\Gamma f)(v) = \gamma{\cdot}v f(v)$.

Proposition 5.1 *The sequence of operators $B^0_V(f)_n$ converges $*$-strongly to a bounded operator $B^0_V(f)$, with $\|B^0_V(f)\| \leq \|f\|_\mathcal{K}$.*

Proof. Let $\{V(v), V(u)\} = F(v) - F(u)$. From the Dirac field anticommutation relations and (5.2) we obtain

$$\begin{aligned}&\left(B_V^0(f)_n - B_V^0(f)_m\right)^* \left(B_V^0(f)_n - B_V^0(f)_m\right)\\ &+ \left(B_V^0(e^{-iF}f)_n - B_V^0(e^{-iF}f)_m\right)\left(B_V^0(e^{-iF}f)_n - B_V^0(e^{-iF}f)_m\right)^*\\ &\qquad = \left(\sum_{i=m+1}^{n} w_i^* w_i\right) \otimes \mathbf{1}\,,\end{aligned}$$

where $w_i = W_{0V}(\overline{e_i}\Gamma f)$. For any vector $\psi \in \mathcal{H}_1 \otimes \mathcal{H}_2$ there is then

$$\begin{aligned}&\|\left(B_V^0(f)_n - B_V^0(f)_m\right)\psi\|^2 + \|\left(B_V^0(e^{-iF}f)_n^* - B_V^0(e^{-iF}f)_m^*\right)\psi\|^2\\ &\qquad = p_\psi\left(\sum_{i=m+1}^{n} w_i^* w_i\right)\,,\end{aligned} \tag{5.3}$$

where $p_\psi(A) := (\psi, A \otimes \mathbf{1}\,\psi)$ agrees for positive A with one of the seminorms defining the σ-weak topology on the space of bounded operators on $\mathcal{H}_1$. We shall show below that

$$\sum_{i=1}^{\infty} w_i^* w_i = \|f\|^2\,\mathbf{1}\,, \tag{5.4}$$

and that the series converges σ-strongly. Hence $B_V^0(f)_n$ and $B_V^0(e^{-iF}f)_n^*$ converge strongly to bounded operators for all f, which implies the $*$-strong convergence of $B_V^0(f)_n$ for all f. Putting $m = 0$ in (5.3) and taking the limit in n we obtain the bound of the norm. To prove (5.4) observe first that for any $x,\ y \in \mathcal{H}_1$ we have $(y, w_i\,x) = \int (y, W_0(V(v))\,x)\,\overline{e_i(v)}\gamma\cdot v f(v)\,d\mu(v) = (e_i, (y, W_0(V(.))x)\,f)_{\mathcal{K}}$, so that

$$\sum_{i=1}^{\infty} |(y, w_i\,x)|^2 = \|(y, W_0(V(.))x) f\|_{\mathcal{K}}^2\,.$$

For fixed x let $\{\varphi_j\}$ be an orthonormal basis of the subspace of $\mathcal{H}_1$ spanned by $W_0(V(v))x$, $v \in H$. Then

$$\begin{aligned}&\sum_{i=1}^{\infty} (x, w_i^* w_i\,x) = \sum_{i,j=1}^{\infty} |(\varphi_j, w_i\,x)|^2\\ &= \sum_{j=1}^{\infty} \int |(\varphi_j, W_0(V(v))x)|^2\,\overline{f(v)}\gamma\cdot v f(v)\,d\mu(v) = \|f\|_{\mathcal{K}}^2\,\|x\|^2\,,\end{aligned}$$

the last equality by the Lebesgue theorem. As $\sum_{i=1}^{n} w_i^* w_i$ is an increasing sequence of positive operators, the above calculation shows that (5.4) holds in the σ-strong sense (e.g. [23], Lemma 2.4.19). □

The building blocks of the representation acting on $\mathcal{H}_1 \otimes \mathcal{H}_2$ are now defined by

$$W(V) = W_0(V) \otimes \mathbf{1}\,, \qquad V \in L\,, \tag{5.5}$$

$$W_V(\chi) = W_{0V}(\chi) \otimes \mathbf{1}\,, \quad V \in \mathcal{R}ad\,, \quad \chi \in \mathcal{S}(H, \mathbf{C})\,, \tag{5.6}$$

$$\begin{aligned} B^\sharp_V(f) = \sum_{i=1}^{\infty} W_{0V}(\overline{e_i}\Gamma f) \otimes B^\sharp(e_i)\,, \\ V \in \mathcal{S}at_e \cup \mathcal{S}at_{-e}\,, f \in \mathcal{S}(H, \mathbf{C}^4)\,. \end{aligned} \tag{5.7}$$

The definition of $B^\sharp_V$ is independent of the choice of the basis $\{e_i\}$, as it is easily shown that

$$(x_1 \otimes y_1, B^\sharp_V(f)\, x_2 \otimes y_2) = \Big(y_1, B^\sharp((x_1, W_0(V(.))x_2)f)\, y_2\Big)\,. \tag{5.8}$$

If D is any sequence of the symbols $W(V)$, W_V and $B^\sharp_V$, and $(f^1, \ldots, f^n)$ form a sequence of type $\flat(D)$, then we define $[D](f^1 \otimes \ldots \otimes f^n)$ as the product of the "building blocks". In view of Prop.5.1 and of the obvious bound $\|W_V(\chi)\| \leq \|\chi\|_{L^1}$, this element is norm continuous in each of f's in the topology of $\mathcal{S}$. By the nuclear theorem it extends then to the function $[D](\chi)$ norm continuous in χ in the $\mathcal{S}$-topology. The conditions (4.12) and (4.13) are satisfied, and the operator norm fulfills the defining conditions of the class $\mathcal{R}$. To complete the proof that we have thus obtained a representation of the algebra $\mathcal{F}$ it remains to show that the relations (4.5–4.11) are satisfied.

The conditions (4.5) and (4.11) are obviously satisfied. It is sufficient to check the other relations for elements $[D](\chi)$ with D's being sequences of two symbols. The relations (4.6) and (4.7) are then quite obvious as well. Eq. (4.8) for $G_V = W_V$ follows from (5.1). For $G_V = B^\sharp_V$ it is easy to show that

$$\begin{aligned} &(x_1 \otimes y_1, \left[W_{V_1} B^\sharp_{V_2}\right](\chi)\, x_2 \otimes y_2) \\ &= \left(y_1, B^\sharp\left(\int (x_1, W_0(V_1(v))W_0(V_2(.))x_2)\, \chi(v, .)\, d\mu(v)\right) y_2\right)\,, \end{aligned}$$

and similarly in the opposite order of symbols, which implies (4.8).

To prove (4.9) and (4.10) we have to make a digression on the extension of products $B^{\sharp_1}(f)B^{\sharp_2}(g)$ in the algebra of the Dirac field. We have mentioned such an extension to the space of Schwartz functions, but now we need a wider family.

Let $\mathcal{K} \otimes \mathcal{K}$ be the tensor product Hilbert space. This space consists of measurable functions $\chi_{\alpha\beta}(v, u)$, for which

$$\sum_{\substack{\alpha,\alpha' \\ \beta,\beta'}} \int \chi^*_{\alpha\beta}(v, u)(\gamma^0\gamma\cdot v)_{\alpha\alpha'}(\gamma^0\gamma\cdot u)_{\beta\beta'}\chi_{\alpha'\beta'}(v, u)\, d\mu(v)\, d\mu(u) < \infty\,.$$

Let, further, $\mathcal{K}\otimes_1\mathcal{K}$ be the subspace of $\mathcal{K}\otimes\mathcal{K}$ consisting of those $\chi\in\mathcal{K}\otimes\mathcal{K}$ for which

$$\|\chi\|_1 := \inf_{\chi=\sum_{i=1}^\infty f_i\otimes g_i} \sum_{k=1}^{\infty} \|f_k\|\|g_k\| < \infty\,.$$

$\|.\|_1$ is a norm on $\mathcal{K}\otimes_1\mathcal{K}$, $\|\chi\| < \|\chi\|_1$ for $\chi\in\mathcal{K}\otimes_1\mathcal{K}$, and $(\mathcal{K}\otimes_1\mathcal{K},\|.\|_1)$ is a Banach space. These statements follow most simply from the following two observations.
(i) $(\mathcal{K}\otimes\mathcal{K},\|.\|)$ is isomorphic with the space of Hilbert-Schmidt operators on $\mathcal{K}$ by the map $\chi\to\mathcal{O}_\chi$, where for $\chi=\sum_{i=1}^\infty f_i\otimes g_i$ the operator $\mathcal{O}_\chi$ is defined by

$$\mathcal{O}_\chi h = \sum_{i=1}^{\infty}(h^c, f_i)g_i\,. \tag{5.9}$$

(ii) Under the same map $(\mathcal{K}\otimes_1\mathcal{K},\|.\|_1)$ is isomorphic with the Banach space of trace class operators on $\mathcal{K}$ [24]. Thus, if $\chi_n\to\chi$ in $\mathcal{K}\otimes_1\mathcal{K}$ then $\chi_n\to\chi$ in $\mathcal{K}\otimes\mathcal{K}$.

Extension of the product of fundamental elements in the Dirac field algebra is now easily achieved. For $\chi=\sum_{i=1}^{n} f_i\otimes g_i$, $f_i, g_i\in\mathcal{K}$, we set $[B^{\sharp_1}B^{\sharp_2}](\chi)=\sum_{i=1}^{n}B^{\sharp_1}(f_i)B^{\sharp_2}(g_i)$. This defines a linear map of the algebraic product $\mathcal{K}\otimes_{\rm alg}\mathcal{K}$ (densly contained in $\mathcal{K}\otimes_1\mathcal{K}$) into the algebra, with the norm bound $\|[B^{\sharp_1}B^{\sharp_2}](\chi)\|\leq\|\chi\|_1$. Hence the map extends to the whole $\mathcal{K}\otimes_1\mathcal{K}$, with the conservation of the bound. For $\chi\in\mathcal{S}$ this reduces to the extension mentioned previously.

The anticommutation relations may be now extended from $\mathcal{K}\otimes_{\rm alg}\mathcal{K}$ to the whole $\mathcal{K}\otimes_1\mathcal{K}$, which gives

$$[B^\sharp B^\sharp](\chi+\chi^{\rm T})=0\,, \tag{5.10}$$

$$[B^0\bar{B}](\chi)+[\bar{B}B^0](\chi^{\rm T})={\rm Tr}\mathcal{O}_\chi\,\mathbf{1}\,, \tag{5.11}$$

where ${\chi^{\rm T}}_{\alpha\beta}(v,u)=\chi_{\beta\alpha}(u,v)$, and ${\rm Tr}\mathcal{O}_\chi$ is the trace of the operator (5.9), ${\rm Tr}\mathcal{O}_\chi=\sum_{i=1}^{\infty}(f_i^c,g_i)$ for $\chi=\sum_{i=1}^{\infty}f_i\otimes g_i$.

After these preparations we take up the proof of the relations (4.9) and (4.10). For x_1, $x_2\in\mathcal{H}_1$ let φ_j be an orthonormal basis of the linear span of vectors $W_0(V_1(v))^*x_1$ and $W_0(V_1(v))x_2$, $v\in H$. Let, further, ψ_λ be an orthonormal basis (not necesserily countable) of $\mathcal{H}_2$, and f, $g\in\mathcal{S}(H,\mathbf{C}^4)$. Expanding $B^{\sharp_2}_{V_2}(g)x_2\otimes y_2$ in the basis $\varphi_j\otimes\psi_\lambda$ and using (5.8) one finds

$$\begin{aligned}&\Big(x_1\otimes y_1, B^{\sharp_1}_{V_1}(f)B^{\sharp_2}_{V_2}(g)x_2\otimes y_2\Big)\\ &=\sum_{i=1}^{\infty}\Big(y_1,[B^{\sharp_1}B^{\sharp_2}]\left((x_1,W_0(V_1(.))\varphi_i)f\otimes(\varphi_i,W_0(V_2(.))x_2)g\right)y_2\Big)\,.\end{aligned}$$

We have

$$\begin{aligned}
&\sum_{i=1}^{\infty} \|(x_1, W_0(V_1(.))\varphi_i) f\|_{\mathcal{K}} \,\|(\varphi_i, W_0(V_2(.))x_2) g\|_{\mathcal{K}} \\
&\leq \left(\sum_{i=1}^{\infty} \|(W_0(V_1(.))^* x_1, \varphi_i) f\|_{\mathcal{K}}^2\right)^{1/2} \left(\sum_{i=1}^{\infty} \|(\varphi_i, W_0(V_2(.))x_2) g\|_{\mathcal{K}}^2\right)^{1/2} \\
&= \|x_1\| \|x_2\| \|f\|_{\mathcal{K}} \|g\|_{\mathcal{K}} \,,
\end{aligned}$$

the last equality by the Lebesgue theorem. Hence, the series

$$\sum_{i=1}^{\infty} (x_1, W_0(V_1(.))\varphi_i) f \otimes (\varphi_i, W_0(V_2(.))x_2) g \tag{5.12}$$

converges both in $\mathcal{K} \otimes_1 \mathcal{K}$ and $\mathcal{K} \otimes \mathcal{K}$, to the same element. The limit in $\mathcal{K} \otimes \mathcal{K}$ is easily found as the point limit of functions, which yields $(x_1, W_0(V_1(v))W_0(V_2(u))x_2) f(v) g(u)$. Thus we have proved that if $\chi \in \mathcal{K} \otimes_{\mathrm{alg}} \mathcal{K}$ then

$$\begin{aligned}
&(x_1, W_0(V_1(.))W_0(V_2(.))x_2)\, \chi \in \mathcal{K} \otimes_1 \mathcal{K} \,, \\
&\|(x_1, W_0(V_1(.))W_0(V_2(.))x_2)\, \chi\|_1 \leq \|x_1\| \|x_2\| \|\chi\|_1 \,,
\end{aligned} \tag{5.13}$$

and for $\chi \in \mathcal{S}(H, \mathbf{C}^4) \otimes_{\mathrm{alg}} \mathcal{S}(H, \mathbf{C}^4) \subset \mathcal{K} \otimes_{\mathrm{alg}} \mathcal{K}$

$$\begin{aligned}
&\Big(x_1 \otimes y_1, [B^{\sharp_1}_{V_1} B^{\sharp_2}_{V_2}](\chi)\, x_2 \otimes y_2\Big) \\
&= \Big(y_1, [B^{\sharp_1} B^{\sharp_2}]\left((x_1, W_0(V_1(.))W_0(V_2(.))x_2)\chi\right)\; y_2\Big) \,.
\end{aligned} \tag{5.14}$$

By continuity (5.13) remains true for $\chi \in \mathcal{K} \otimes_1 \mathcal{K}$, and (5.14) for $\chi \in \mathcal{S}(H^2, \mathbf{C}^{4^2}) \subset \mathcal{K} \otimes_1 \mathcal{K}$. Denote $\chi'_{\alpha\beta}(v,u) = e^{(i/2)\{V_1(v), V_2(u)\}} \chi_{\alpha\beta}(v,u)$ and $\chi''_{\alpha\beta}(v,u) = e^{(i/2)\{V_2(u), V_1(v)\}} \chi^{\mathrm{T}}_{\alpha\beta}(v,u)$. Using the Weyl relations for W_0 one obtains from (5.14)

$$\begin{aligned}
&\Big(x_1 \otimes y_1, \Big([B^{\sharp_1}_{V_1} B^{\sharp_2}_{V_2}](\chi') + [B^{\sharp_2}_{V_2} B^{\sharp_1}_{V_1}](\chi'')\Big)\, x_2 \otimes y_2\Big) \\
&= \Big(y_1, \Big([B^{\sharp_1} B^{\sharp_2}](\Psi) + [B^{\sharp_2} B^{\sharp_1}](\Psi^{\mathrm{T}})\Big)\, y_2\Big) \,,
\end{aligned}$$

where $\Psi_{\alpha\beta}(v,u) = (x_1, W_0(V_1(v) + V_2(u))x_2)\, \chi_{\alpha\beta}(v,u)$. The relation (4.9) follows now from (5.10). The proof of (4.10) will be complete by (5.11) and the definition (5.6) if we show that

$$\mathrm{Tr}\mathcal{O}_\Psi = \int (x_1, W_0(V_1(v) + V_2(v))\, x_2) \sum_{\alpha,\beta} (C^{-1}\gamma \cdot v)_{\alpha\beta}\, \chi_{\alpha\beta}(v,v)\, d\mu(v) \,.$$

The rhs may be written as

$$\int (x_1, W_0(V_1(v)) W_0(V_2(v))\, x_2) \sum_{\alpha,\beta} (C^{-1}\gamma \cdot v)_{\alpha\beta}\, \chi'_{\alpha\beta}(v,v)\, d\mu(v) \,.$$

This formula defines a distribution on $\chi' \in \mathcal{S}$, so it is sufficient to take $\chi' = f \otimes g$, f, $g \in \mathcal{S}(H, \mathbf{C}^4)$. Then Ψ is given by (5.12) and

$$\begin{aligned}\mathrm{Tr}\mathcal{O}_\Psi &= \sum_{i=1}^{\infty} \Big(\overline{(x_1, W_0(V_1(.))\varphi_i)} f^c, (\varphi_i, W_0(V_2(.))x_2) g \Big) \\ &= \sum_{i=1}^{\infty} \int (x_1, W_0(V_1(.))\varphi_i)\, (\varphi_i, W_0(V_2(.))x_2)\, \overline{f^c(v)} \gamma \cdot v g(v)\, d\mu(v)\,,\end{aligned}$$

which yields the desired relation by the Lebesgue theorem. This ends the proof of the conditions of our algebra.

6 Discussion and outlook

We have shown how heuristic quantization of the asymptotic structure of classical field electrodynamics leads to the construction of an asymptotic algebra of fields, whose states may be expected to describe the structure of collision states in quantum electrodynamics, including the charged states. This algebra is a C^*-algebra, so there is no need for the indefinite metric formalism. The charged fields are accompanied by Coulomb fields, which solves the problem of Gauss' law in charged states. The construction depends on the choice of a class of "satellite" fields. This choice was left open to some extent. It corresponds to selecting various "clouds" of free radiation field accompanying the particle in addition to the Coulomb field. Three classes of satellite fields satisfying the defining conditions (i)–(iv) in Sec.4 are worth mentioning:
(a) the class of all fields satisfying the conditions (i)–(iv);
(b) the subclass consisting of fields of the form $V_a(v; s, l) = V_{ea}(v, l) + V_{0a}(s, l)$, where $V_{ea}(v, l)$, defined in (i), corresponds to the Coulomb field, and $V_{0a} \in L$;
(c) the subclass of fields for which $V_a(v; -\infty, l)$ does not depend on v.
The first choice is the most general one within the limits of our construction. The second possibility is the simplest one. In that case the smeared Weyl elements reduce to the (simple) Weyl elements, as the space $\mathcal{R}ad$ is then naturally isomorphic with L. We mention as an aside that an explicit faithful representation of the corresponding algebra can be constructed. The choice (c) has a clear physical interpretation: the long-range tail of the cloud accompanying the particle is chosen to compensate the velocity dependence of the tail of the Coulomb field. The total satellite field has then a velocity independent flux at spatial infinity.

The last possibility seems to be the one closest to the picture emerging from the analysis of superselection sectors structure in the algebraic framework of local observables. However, we leave the problem of specifying the physically justified choice of satellite fields open at present.

The solution of this problem may depend on the answer to the important question of whether our algebra may be obtained by some limiting procedure from the field algebra of the full theory. This seems a difficult problem, but the classical theory gives hints, how such a limiting process could look like. The quantum version of the null infinity limit of the electromagnetic field may be thought of as an LSZ-type limit in lightlike directions, which brings in mind the construction of Buchholz [3]. For the matter field one would have to choose a gauge in a class supplying a quantum analog of the class mentioned in Sec.2. An LSZ-type limit on the hyperboloid $x^2 = \lambda^2$, $x^0 > 0$ with $\lambda \to \infty$ may then be expected to exist.

Important as the latter problem may be, the asymptotic algebra also deserves further investigations on its own. The following physically interesting problems may be posed within this framework.
(i) How can the local observables be characterized and what is their relation to the nonlocal ones? That the latter observables are present may be read off from the properties of the symplectic form.
(ii) The Poincaré group acts naturally on the algebra as a group of automorphisms. Do there exist irreducible Poincaré-covariant representations of the algebra? If so, which superselection structure of the algebra of local observables is implied? (We recall that in charged superselection sectors the Lorentz group has to be spontaneously broken).
(iii) What is the measure class of the spectrum of energy-momentum in translation-covariant, positive energy representations (infraparticle problem)?

Finally, let us mention for completeness that the construction given in this paper may be reflected in time, yielding the asymptotic "in"-algebra. If these two algebras fit into the interacting theory, the scattering problem may be considered. One may hope, for instance, that the perturbation calculus in a suitable gauge, starting from the quasi-free theory supplied by our algebra, should be infrared-regular.

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Appendix Equivalence of spinor and tensor formulas

We prove here the equivalence of tensor and spinor versions of those formulas which appear in Sec.2, but were given only in the spinor form in [16].

For $l_a = o_A o_{A'}$ in the notation of [16], $\partial_A = \partial/\partial o^A$ and $\partial_a = \partial/\partial l^a$, one has $\partial_A \alpha(l) = o^{A'}\partial_{AA'}\alpha(l)$, hence

$$(l_a\partial_b - l_b\partial_a)\alpha(l) = -(\epsilon_{A'B'}o_{(A}\partial_{B)} + \epsilon_{AB}o_{(A'}\partial_{B')})\alpha(l)\,, \tag{A.1}$$

and the integral identity (2.10) is then equivalent to (A8) of [16].

The electromagnetic field tensor and spinor are connected by $F_{ab} = \epsilon_{A'B'}\varphi_{AB} + \epsilon_{AB}\overline{\varphi}_{A'B'}$, hence the field $\varrho_{AA'}(x) = \varphi_{AB}(x)x^B_{A'}$ ((2.28) in [16]) is equivalently expressed as $\varrho_a(x) = x^b\,{}^-F_{ba}(x)$, where ${}^-F_{ba}$ is the anti-selfdual part of F_{ba}. Its null asymptotic $\lim_{R\to\infty} R\varrho_a(x+Rl) = N_a(x{\cdot}l, l)$ is given by $N_a(s,l) = \partial_{A'}\zeta_A(s,l)$, where $\zeta_A(s,l) = o_{A'}V_A^{A'}$, $V_a(s,l)$ defined in (2.1) (see [16], Eq.(2.44)). We have

$$\partial_{A'}\zeta_A = -V_a + o_{B'}\partial_{A'}V_A^{B'} = -V_a - \frac{1}{2}\epsilon_{B'A'}o^{C'}\partial_{C'}V_A^{B'} + o_{(B'}\partial_{A')}V_A^{B'}\,.$$

In view of homogeneity (2.2) and using (A.1) this may be written as

$$N_a(s,l) = \frac{1}{2}(s\dot{V}_a(s,l) - V_a(s,l)) + {}^+(l_a\partial_b - l_b\partial_a)V^b(s,l)\,,$$

where ${}^+(l_a\partial_b - l_b\partial_a)$ is the selfdual part of $l_a\partial_b - l_b\partial_a$. It was shown in [16] (and may be also shown without use of spinors) that from (2.2) and (2.3) now follows that the limit values

$N_a(\pm\infty, l)$ are proportional to l_a, that is

$$-\frac{1}{2}V_a(+\infty,l) + {}^+(l_a\partial_b - l_b\partial_a)V^b(+\infty,l) = -l_a q(l)\,,$$
$$-\frac{1}{2}V_a(-\infty,l) + {}^+(l_a\partial_b - l_b\partial_a)V^b(-\infty,l) = -l_a \kappa(l)\,,$$

which corresponds with the equations (2.54) and (2.56) of [16] (there we used $\sigma = \kappa - q$ instead of κ). The conditions of reality of q and κ ((3.32) in [16]), and the above equations in that case are now equivalent to (2.5), and (2.6) and (2.7) respectively.

It was shown in [16] (Eq.(2.64)) that

$$o^{A'}{V^{\mathrm{out}}}_{A'A}(-\infty,l) = \partial_A\Phi(l) \tag{A.2}$$

for some $\Phi(l)$ homogeneous of degree 0. In view of (A.1) this is equivalent to (2.14). The gauge of ${V^{\mathrm{out}}}_a(-\infty,l)$ (2.17) is written with the use of (A.1) as

$${V^{\mathrm{out}}}_a(-\infty,l) = \partial_A(o^B g_{BA'}(l)) + \partial_{A'}(o^{B'}g_{B'A}(l))\,.$$

As $o^B g_{BA'}(l)$ are in $1:1$ correspondence with G_{ab}, the gauge has the form ${V^{\rm out}}_a(-\infty,l) = \partial_A h_{A'}(o,\bar{o}) + \partial_{A'}\overline{h}_A(o,\bar{o})$, where $h_{A'}(\alpha o,\overline{\alpha}\bar{o}) = \overline{\alpha}^{-1}h_{A'}(o,\bar{o})$, $h_{A'}(o,\bar{o})o^{A'} = \Phi(l)$, and in the statements (i)–(iii) following (2.17) the tensor G_{ab} may be replaced by $h_{A'}(o,\bar{o})$ satisfying these conditions. This representation satisfies (A.2), so (i) is proved. We get a special gauge choosing

$$h_{A'}(o,\bar{o}) = \frac{t_{A'A}o^A}{t{\cdot}l}\Phi(l)\,,$$

where t is any unit, positive timelike vector. Any other gauge differs by $\beta(l)l_a$, $\beta(l)$ homogeneous of degree -2. Let $\alpha(l) = \beta(l) - c\,(t{\cdot}l)^{-2}$, with such a constant c that $\int\alpha(l)\,d^2l = 0$. There exists then a homogeneous of degree 0 function $A(l)$, such that $\partial_A\partial_{A'}A(l) = \frac{1}{2}o_A o_{A'}\alpha(l)$. $A(l)$ is determined up to an additive constant. The new gauge is then determined by

$$h'_{A'}(o,\bar{o}) = \frac{t_{A'A}o^A}{t{\cdot}l}(\Phi(l)+c) + \partial_{A'}A(l)\,.$$

With this formula the statements (ii) and (iii) following (2.17) are seen to be true.

Finally, the angular momentum term ΔM_{ab} (2.22) is the tensor version of the angular momentum spinor term $\Delta\mu_{AB} = \dfrac{1}{2\pi}\displaystyle\int q o_{(A}\partial_{B)}\Phi\,d^2l$, easily obtained from $\Delta M_{ab} = \epsilon_{A'B'}\Delta\mu_{AB} + \epsilon_{AB}\overline{\Delta\mu}_{A'B'}$ by the use of (A.1).

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